CONNECTICUT STATE ASSOCIATION OF MATH LEAGUES
State Math Match, 2025: Solutions

1-1 121% of $\left(\frac{1}{3}+\frac{2}{5}\right)^{-3}$ is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.
[Answer: 179]

Solution:

$$121\%\ of\ \left(\frac{1}{3}+\frac{2}{5}\right)^{-3} = \frac{121}{100}\left(\frac{11}{15}\right)^{-3} = \frac{121}{100}\left(\frac{15}{11}\right)^{3} = \frac{11^2}{2^2(5^2)} \times \frac{3^3(5^3)}{11^3} = \frac{3^3(5)}{2^2(11)} = \frac{135}{44}$$

$$135+44=179$$

1-2 Find the smallest positive integer n such that $n!\cdot(n-1)!\cdot(n-2)!\cdot\cdots\cdot 1!$ is divisible by 2^{56}.
[Answer: 12]

Solution:

$2!$ *has* 2 *as a factor* 1 *time*
$3!$ *has* 2 *as a factor* 1 *time*
$4!$ *has* 2 *as a factor* 3 *times*
$5!$ *has* 2 *as a factor* 3 *times*
$6!$ *has* 2 *as a factor* 4 *time*
$7!$ *has* 2 *as a factor* 4 *time*
$8!$ *has* 2 *as a factor* 7 *time*
$9!$ *has* 2 *as a factor* 7 *time*
$10!$ *has* 2 *as a factor* 8 *time*
$11!$ *has* 2 *as a factor* 8 *time*
$12!$ *has* 2 *as a factor* 10 *time*

1-3 Two ordinary, fair dice are thrown repeatedly. On each throw, the "score" is the sum of the numbers on the two uppermost faces. The probability that the first score of 4 is obtained earlier in the sequence of scores than the first score of 7 and the first score of 7 is obtained earlier than the first score of 5 is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.
[Answer: 74]

Solution:
For the sake of clarity, we will assume that one of the dice is blue and the other is red. Let (x,y) represent a score of x on the blue die and y on the red die. On any given throw, the possible outcomes are $(1,1),(1,2),\ldots,(6,6)$.

We will refer to outcomes that result in a score of 4, 7, or 5 as "interesting" outcomes. Further, we will refer to outcomes that give a score of 4 as being "X-type" outcomes, those that give a score of 7 as being "Y-type" outcomes, and those that give a score of 5 as being "Z-type" outcomes.

We require the probability that the first interesting outcome is an X-type outcome and the next Y- or Z-type outcome is a Y-type outcome.

There are 3 X-type outcomes ($(1,3),(2,2)$, and $(3,1)$), 6 Y-type outcomes, and 4 Z-type outcomes, making a total of 13 interesting outcomes, and each one of these interesting outcomes is equally likely to be the first interesting outcome to appear. So, the probability that the first interesting outcome is an X-type outcome is 3/13.

We now assume that this event has occurred, and find the probability that the first Y- or Z-type outcome after that is a Y-type outcome. There are 10 Y- or Z-type outcomes, and each of those 10 outcomes is equally likely to be the next Y- or Z-type outcome. So, the probability that the next Y- or Z-type outcome is a Y-type outcome is 6/10.

Thus, the required probability is $\left(\frac{3}{13}\right)\left(\frac{6}{10}\right) = \frac{9}{65}$, and the answer is 74.

2-1 For any real, positive value of x, $\sqrt{x \cdot \sqrt[3]{x \cdot \sqrt[4]{x}}} = x^{p/q}$, where p and q are relatively prime positive integers. Find $p + q$.
[Answer: 41]

Solution:

$$\left(x^1\left(\left(x^1\left(x^{\frac{1}{4}}\right)\right)^{\frac{1}{3}}\right)\right)^{\frac{1}{2}} = \left(x^1\left[\left(x^{\frac{5}{4}}\right)^{\frac{1}{3}}\right]\right)^{\frac{1}{2}} = \left(x^1\left(x^{\frac{5}{12}}\right)\right)^{\frac{1}{2}} = x^{\frac{17}{24}}$$

17 + 24 = 41

2-2 Find the <u>sum</u> of the <u>squares</u> of all real values of x such that

$$\frac{|x|}{1+|x|} = \frac{6+x}{1-x}$$

[Answer: 9]

Solution:

$\frac{|x|}{1+|x|} = \frac{6+x}{1-x}$, *if* $x > 0$ $\frac{x}{1+x} = \frac{6+x}{1-x}$ *results in* $1x - x^2 = x^2 + 7x + 6$

hence $2x^2 + 6x + 6 = 0$ *has no real solutions*

if $x < 0$, $|x| = -x$ *thus* $\frac{-x}{1-x} = \frac{6+x}{1-x}$, $-2x = 6$, $x = -3$ *is the only solution* $(-3)^2 = 9$

2-3 Find the number of integer values of n, $1 \le n \le 2025$, such that the following inequality has exactly two integer solutions.

$$\left|\frac{nx}{2025} - 1\right| < \frac{n}{2025}$$

[Answer: 2010]

Solution:
Note that the given inequality is equivalent to $\left|x - \frac{2025}{n}\right| < 1$. So, we require the number of values of n such that there are exactly two integers within 1 unit of $\frac{2025}{n}$. If $\frac{2025}{n}$ is an integer, there are no integers within 1 unit of $\frac{2025}{n}$. If $\frac{2025}{n}$ is not an integer, then there are exactly two integers within 1 unit of $\frac{2025}{n}$. (For example, if $n = 7$, then $\frac{2025}{n} = 289\frac{2}{7}$, and the integers 289 and 290 are within 1 unit of $\frac{2025}{n}$.) So, we require the number of values of n, $1 \le n \le 2025$, such that n is not a divisor of 2025.

Now, $2025 = 3^4 \cdot 5^2$, so the number of divisors of 2025 is $5 \cdot 3 = 15$. So, the number of values of n that are <u>not</u> divisors of 2025 is $2025 - 15 = 2010$.

3-1 A pyramid has a square base with side length 10, and the height of the pyramid is 12. When the base of the pyramid is placed on a horizontal surface, the vertex of the pyramid lies directly above the center of the base. Find the total surface area of the pyramid, including the base.
[Answer: 360]

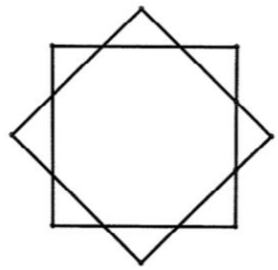
Solution:
The distance from the midpoint of the side of the square base is 5 resulting in a 5, 12 13 Pythagorian triplet, slant height is 13.

$4\ triangular\ side\ :\ 4\left(\frac{1}{2}(10)(13)\right) = 260$

$Area\ of\ the\ square\ base\ \left(10^2\right) = 100$

$Total\ surface\ area\ =\ 260 + 100 =\ 360$

3-2 As shown in the diagram below, two congruent squares overlap in such a way that their intersection points form a regular octagon.



If the side length of the original squares is 2, then the area of the octagon is $a\sqrt{b}+(-1)^n c$, where a, b, and c are positive integers, n is 0 or 1, and b is not divisible by the square of any prime number. Find $a+b+c+n$.
[Answer: 19]

<u>Solution:</u>

Using the diagonal of the square minus the length of the square gives you the length of each side of the of the regular octagon:

$2\sqrt{2}$ *is the length of the diagonal of square with side* 2

$2\sqrt{2}-2$ *is the length of the sides of the regular octagon*

Dissecting the octagon into two isosceles trapezoids and one middle rectangle calculates the area:

Height of the trapezoids is the length of the side of octagon divided by $\sqrt{2}$:

$\frac{(2\sqrt{2}-2)}{\sqrt{2}} = 2-\sqrt{2} =$ *height*

with bases of $2\sqrt{2}-2$ (*the side of the octagon*) *and* 2 (*length of the side of the square*)

$2\left(\frac{1}{2}\left(2\sqrt{2}-2+2\right)\left(2-\sqrt{2}\right)\right) = 4\sqrt{2}-4$ *area of two trapezoids*

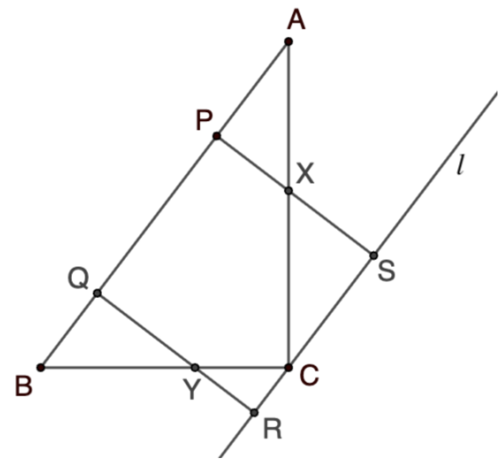
$Area\ of\ rec\tan gle\ :\ 2\left(2\sqrt{2}-2\right) = 4\sqrt{2}-4$

$Total\ area\ \left(4a\sqrt{2}-4\right)+\left(4\sqrt{2}-4\right) = 8\sqrt{2}-8$

$a=8,\ b=2,\ n=1,\ c=8,$

$8+2+1+8=19$

3-3 In right triangle ABC with right angle C, line l is drawn through C and is parallel to $\overline{AB}$, as shown in the diagram below. Points P and Q lie on $\overline{AB}$ with P between A and Q, and points R and S lie on l with C between R and S such that $PQRS$ is a square. Let $\overline{PS}$ intersect $\overline{AC}$ in X, and let $\overline{QR}$ intersect $\overline{BC}$ in Y. The inradius of triangle ABC is 10, and the area of square $PQRS$ is 576. Find the sum of the inradii of triangles AXP, CXS, CYR, and BYQ.



[Answer: 14]

Solution

Let $BC = a$ and $AC = b$. Then $AB = \sqrt{a^2 + b^2}$.

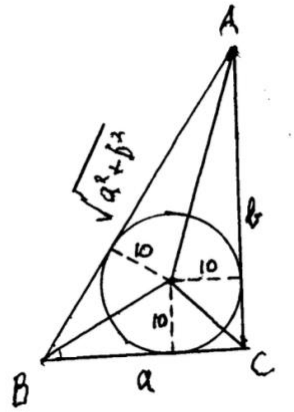
Let the inradii of triangles AXP, CXS, YCR, YBQ be r_1, r_2, r_3, r_4, respectively, and note that these four triangles are all similar to triangle ABC. Consequently, the inradii of all five triangles are in proportion to their hypotenuse lengths.

Thus, $\frac{r_1}{10} = \frac{AX}{AB}$ and $\frac{r_2}{10} = \frac{XC}{AB}$. Adding, we have $\frac{r_1+r_2}{10} = \frac{AX+XC}{AB} = \frac{AC}{AB} = \frac{b}{\sqrt{a^2+b^2}}$.

Similarly, $\frac{r_3+r_4}{10} = \frac{a}{\sqrt{a^2+b^2}}$.

So, $r_1 + r_2 + r_3 + r_4 = \frac{10(a+b)}{\sqrt{a^2+b^2}}$. (1)

Note that $PQ = QR = RS = SP = \sqrt{576} = 24$. This is the height of triangle ABC when its base is AB, so the area of triangle ABC can be written as $\frac{1}{2}\sqrt{a^2+b^2} \cdot 24$.



Referring to the diagram above, the area of triangle *ABC* can also be written as $\frac{1}{2}a \cdot 10 + \frac{1}{2}b \cdot 10 + \frac{1}{2}\sqrt{a^2+b^2} \cdot 10$.

Thus, $\frac{1}{2}\sqrt{a^2+b^2} \cdot 24 = \frac{1}{2}a \cdot 10 + \frac{1}{2}b \cdot 10 + \frac{1}{2}\sqrt{a^2+b^2} \cdot 10$,

So, $24 = \frac{10(a+b)}{\sqrt{a^2+b^2}} + 10$. (2)

From (1) and (2) we get that $r_1 + r_2 + r_3 + r_4 = \frac{10(a+b)}{\sqrt{a^2+b^2}} = 24 - 10 = 14$.

4-1 The first term of an infinite geometric series is 8, and the sum to infinity of the series is 12. The fourth term of the series is $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a+b$.
[Answer: 35]

Solution:
$\frac{8}{1-r} = 12$, $r = \frac{1}{3}$, *the 4th term is* $\frac{8}{27}$
$8 + 27 = 35$

4-2 Suppose that $4^{1/x} - 8^{1/y} = 0$ and $\log_2 x - \log_4 y = 0$. Then $x + y = \frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
[Answer: 19]

Solution:

$4^{\frac{1}{x}} - 8^{\frac{1}{y}} = 0,\ 4^{\frac{1}{x}} = 8^{\frac{1}{y}}$

$2^{\frac{2}{x}} = 2^{\frac{3}{y}},\ \frac{2}{x} = \frac{3}{y},\quad 2y = 3x$

Using $\log_2 x$ - $\log_4 y$ =0 gives $\log_2 x = \log_4 y$.
Let a=$\log_2 x$, $2^a = x$ and a=$\log_4 y$, $4^a = y$, 2^{2a}=y

$\frac{2}{2^a} = \frac{3}{2^{2a}}$ *gives* $\frac{2(2^{2a})}{2^a} = 3$ *thus* $2^{a+1} = 3$

subing in x for 2^a *gives* $2x = 3,\ x = \frac{3}{2}$

$2y = 3x,\ y = \frac{9}{4}$

$\frac{3}{2} + \frac{9}{4} = \frac{15}{4},\ 15 + 4 = 19$

4-3 Let $P(x)$ be a fifth-degree polynomial with integer coefficients such that the equation $P(x) = 0$ has at least one integer solution. Suppose that $P(2) = 13$ and $P(10) = 5$. Find the value of x that *must* satisfy the equation $P(x) = 0$.
[Answer: 15]

Solution:
Let k be an integer solution of the equation $P(x) = 0$. Then $P(x) = (x - k)q(x)$, where q is a degree-four polynomial. Note that the coefficients in the polynomial $q(x)$ are integers. (You can convince yourself of this by imagining doing the division $P(x)/(x - k)$ using synthetic or long division.) So, using the information given, $13 = P(2) = (2 - k)q(2)$ and $5 = P(10) = (10 - k)q(10)$. Note that all factors in these equations are integers, so $2 - k$ divides 13 and $10 - k$ divides 5. From the first of these observations we get that $2 - k = -13, -1, 1$ or 13, so $k = 15, 3, 1$ or -11. From the second observation we get that $10 - k = -5, -1, 1$ or 5, so $k = 15, 11, 9$ or 5. Thus the only possibility for k is 15, meaning that 15 is a solution of the equation $P(x) = 0$.

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5-1 Parabola 1 has equation $y = x^2 - 6x + 10$ and Parabola 2 has equation $y = 2x^2 + 3x$. The parabolas intersect at point P in the first quadrant. Let the distance of P from the vertex of Parabola 1 be d. Find d^2.
[Answer: 20]

Solution:

$Parabola\ 1:\ y = x^2 - 6x + 10,\ \ y = (x-3)^2 + 1,\ vertex\ (3,1)$

$x^2 - 6x + 10 = 2x^2 + 3x$

$x2 + 9x - 10 = 0\ ,\ \cancel{x = -10}\ or\ x = 1\ first\ quadrant\ gives\ (1,5)\ as\ the\ pt\ of\ int$

$d^2 = (3-1)^2 + (1-5)^2 = 4 + 16 = 20$

5-2 The center of a circle lies on the y-axis, and the circle passes through the points $(7,-1)$ and $(-3,-3)$. The circle intersects the positive x-axis at $(h,0)$. Find h^2.
[Answer: 66]

Solution:

Using that the distance formula from the center (0,k) and the two points (7,-1) and (-3, -3) is equal gives:

$(-3-0)^2 + (-3-k)^2 = (7-0)^2 + (-1-k)^2$

$9 + \left(9 + 6k + k^2\right) = 49 + \left(1 + 2k + k^2\right)$

$18 + 6k = 50 + 2k,\ \ 4k = 32,\ \ k = 8$

$Center: (0,\ 8)\ \ r^2 = (-3-0)^2 + (-3-8)^2 = 130$

$Subbing\ in\ (h,0),\ (h-0)^2 + (0-8)^2 = 130$

$h^2 + 64 = 130,\ \ h^2 = 66$

5-3 The line $y = mx + b$ is tangent to the curve $y = x^4 + 4x^3 - 16x^2 + 6x - 5$ at two points. Find $m - b$.
[Answer: 151]

Solution:
Let $f(x) = x^4 + 4x^3 - 16x^2 + 6x - 5$ and $g(x) = mx + b$. Since the graph of $y = g(x)$ is tangent to the graph of $y = f(x)$ at two points, the polynomial equation $f(x) - g(x) = 0$ has two repeated solutions. So $f(x) - g(x)$ takes the form $(x-\alpha)^2(x-\beta)^2 = \left((x-\alpha)(x-\beta)\right)^2$. So $f(x) - g(x)$ is the square of a quadratic polynomial. That is, $x^4 + 4x^3 - 16x^2 + (6-m)x - (5+b) = (x^2 + px + q)^2$, where p and q are constants. Equating the coefficients of x^3 tells us that $p = 2$ and then equating the coefficients of x^2 tells us that $q = -10$. Thus, $x^4 + 4x^3 - 16x^2 + (6-m)x - (5+b) = (x^2 + 2x - 10)(x^2 + 2x - 10)$. Equating coefficients of x tells us that $6 - m = -40$ and equating constant terms tells us that $-5 - b = 100$. So $m = 46$ and $b = -105$, making $m - b = 151$.

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6-1 Let $z = 1 + 3i$. Then $\left(3z + \frac{10}{z}\right)^2 = a + bi$, where a and b are real. Find $|a| + |b|$.
[Answer: 68]

Solution:
$z = 1 + 3i$ *find* $\left(3z + \frac{10}{z}\right)^2$ *simplify* $\frac{10}{1+3i}\left(\frac{1-3i}{1-3i}\right) = \frac{\cancel{10}(1-3i)}{\cancel{10}}$
$(3(1+3i) + 1 - 3i)^2 = (3 + 9i + 1 - 3i)^2 = (4 - 6i)^2$
$16 + 48i - 36 = -20 + 48i, \quad |-20| + |48| = 68$

6-2 Suppose that

$$\tan\theta = \frac{5\cos\theta}{3\sin\theta - 2\cos\theta}$$

Then the <u>sum</u> of the <u>squares</u> of the possible values of $\tan\theta$ is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
[Answer: 43]

Solution:
Use sin a = y/r, cos a =x/r and tan a = y/x substitution:
$\tan\theta = \frac{5\cos\theta}{3\sin\theta - 2\cos\theta}$
$\frac{y}{x} = \frac{5x}{3y-2x}, \quad 5x^2 = 3y^2 - 2xy$
$5x^2 + 2xy - 3y^2 = 0, \quad (5x - 3y)(1x + y) = 0$
$\frac{y}{x} = \frac{5}{3}$ *or* $-1, \ \left(\frac{5}{3}\right)^2 + (-1)^2 = \frac{25}{9} + 1 = \frac{34}{9}$
$34 + 9 = 43$

6-3 Let $S = \sin^2 4° + \sin^2 8° + \sin^2 12° + \cdots + \sin^2 176°$.
Then $S = \frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.
[Answer: 47]

Solution:
From the formula $\cos 2\theta = 1 - 2\sin^2\theta$ it follows that $\sin^2\theta = (1 - \cos 2\theta)/2$.
Thus,

$$S = \frac{1 - \cos 8°}{2} + \frac{1 - \cos 16°}{2} + \cdots + \frac{1 - \cos 352°}{2} \ .$$

The arithmetic sequence 8, 16, … , 352 has 44 terms, so

$$S = \frac{44}{2} - \frac{1}{2}(\cos 8° + \cos 16° + \cdots + \cos 352°)$$

Now note that $\cos 0° + \cos 8° + \cos 16° + \cdots + \cos 352°$
$= \mathrm{Re}(\mathrm{cis}\, 0° + \mathrm{cis}\, 8° + \cdots + \mathrm{cis}\, 352°)$, where $\mathrm{cis}\,\theta$ denotes $\cos\theta + i\sin\theta$. Note, further, that the complex numbers $\mathrm{cis}\, 0$, $\mathrm{cis}\, 8°, \ldots, \mathrm{cis}\, 352°$ are equally spaced around the origin in the complex plane, so $\mathrm{cis}\, 0° + \mathrm{cis}\, 8° + \cdots + \mathrm{cis}\, 352° = 0$. Thus,
$\cos 0° + \cos 8° + \cos 16° + \cdots + \cos 352° = 0$.

It follows that $\cos 8° + \cos 16° + \cdots + \cos 352° = -\cos 0 = -1$.

So

$$S = \frac{44}{2} - \frac{1}{2}(-1) = \frac{45}{2},$$

and the answer to the question is $45 + 2 = 47$.

T-1 Suppose that $\frac{1}{n}$ is added to $\frac{1}{100}$ and the sum is expressed as a fully reduced fraction. For how many values of n selected from the set $\{1, 2, 3, \ldots, 99\}$ is the denominator of the sum equal to $100n$?
[Answer: 40]

Solution:
$\frac{1}{x} + \frac{1}{100} = \frac{100 + x}{100x}$. If n divides 100x, it must divide 100 and x for the fraction to be reducible. This is true if and only if 100 and x are relatively prime. So x cannot be a multiple of 2, 4, or 5. There are 49 even numbers and 10 multiples of 5, which are not also multiples of 2. 99 – 59 = 40.

T-2 The Twins-R-Us day care center has 35 sets of twins enrolled, and these are the only children enrolled in the program. Of the children enrolled in the program, 38 are boys, and there are four more sets of girl-girl twins than of girl-boy twins. How many sets of boy-boy twins are enrolled?
[Answer: 15]

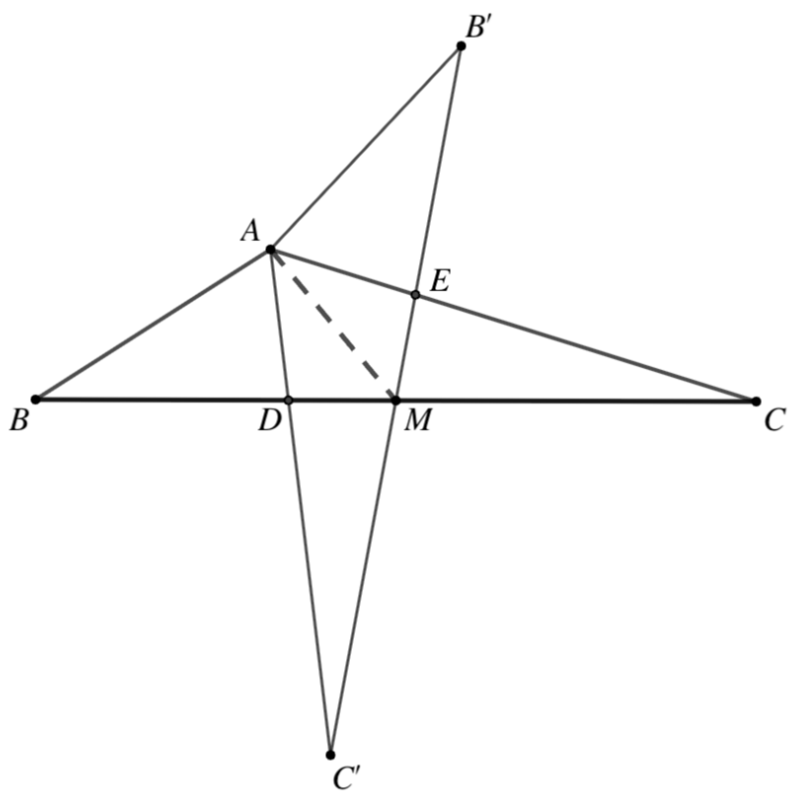
Solution:
Let

g = # of sets of girl twins,	1) g+b+c=35	4) g-4 = c
b = # of sets of boy twins,	2) 2b+c=38	
c = # of sets of girl, boy twins	3) 2g+c=32 using 2) - 3) b= g+3	

By substitution: g+ g+3 +g-4 =35, 3g=36, g=12, b=15

T-3 [Terminology: The "reflection of point P in line l" is a point P' (called the image of P) such that l is the perpendicular bisector of the line segment $\overline{PP'}$. The reflection of a set of points is the set of their images.]

Triangle ABC is reflected in its median $\overline{AM}$ (extended), as shown in the diagram below. The images of points B and C are B' and C', respectively. Let D be the point of intersection of $\overline{BC}$ and $\overline{AC'}$ and let E be the point of intersection of $\overline{AC}$ and $\overline{B'C'}$. Suppose that $AE = 6$, $EC = 12$, and $BD = 10$. Then $AB = p\sqrt{q}$, where p and q are positive integers and q is not divisible by the square of any prime number.
Find $p + q$.



[Answer: 11]

Solution:
First note that $AD = AE = 6$. Now note that line *AM* bisects angle *DAC*. Thus, using the angle bisector theorem in triangle ADC, $DM/MC = AD/AC = 6/18 = 1/3$. So, if we let $DM = x$, then $MC = 3x$. Now note that *M* is the midpoint of line *BC*. Thus, $10 + x = 3x$, so $x = 5$. Therefore, $DC = x + 3x = 20$.

Using the law of cosines on triangle *ADC*,

$$\cos(\angle ADC) = \frac{6^2 + 20^2 - 18^2}{2 \cdot 6 \cdot 20} = \frac{7}{15}.$$

So $\cos(\angle ADB) = -7/15$. Thus, using the law of cosines on triangle *ADB*,

$$AB^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10\left(-\frac{7}{15}\right) = 192 = 64 \cdot 3$$

So $AB = 8\sqrt{3}$, and the answer to the question is $8 + 3 = 11$.

T-4 Let

$$(x^3+ax^2+2x-5)^{19}(x^2+bx-41)^8(x^4-x^3+x-7)^6$$
$$= x^{97}+391x^{96}+c_{95}x^{95}+c_{94}x^{94}+\cdots+c_1x+c_0$$

be an identity, where $a, b, c_{95}, c_{94}, \ldots, c_1, c_0$ are integers, and $a+b<10$. Find the smallest possible value for a.
[Answer: 31]

Solution:
Consider, first, the factor $(x^3+ax^2+2x-5)^{19}$. Think of it as the product consisting of the factor (x^3+ax^2+2x-5) written 19 times. The first term of the product is x^{57}. To get the term in x^{56} you take x^3 from eighteen of the factors and ax^2 from the other factor. This can be done nineteen different ways, so the term in x^{56} is $19ax^{56}$. Similarly, $(x^2+bx-41)^8 = x^{16}+8bx^{15}+\cdots$ and $(x^4-x^3+x-7)^6 = x^{24}-6x^{23}+\cdots$.

So,

$$(x^3+ax^2+2x-5)^{19}(x^2+bx-41)^8(x^4-x^3+x-7)^6$$
$$= (x^{57}+19ax^{56}+\cdots)(x^{16}+8bx^{15}+\cdots)(x^{24}-6x^{23}+\cdots).$$

The term in x^{96} in the product of these three factors is obtained by combining the terms with the highest exponents in each of two of the factors with the term with the second-highest exponent in the third factor, this being done three different ways. We obtain the coefficient of x^{96} to be $19a+8b-6$. Thus, using the information given in the question, $19a+8b-6=391$, so $19a+8b=397$, and remember that a and b are integers. Call this equation (1).

To reduce the size of the numbers involved, define new integer variables p and q by $a=p+20$ and $b=q+2$. Then, $19(p+20)+8(q+2)=397$. So $19p+8q=1$. We search for a solution to this equation by trying successive values of p, and for each value of p seeing whether a solution for q exists. We find that $p=3, q=-7$ is a solution. Thus, $a=23, b=-5$ is a solution to equation (1).

Notice, in equation (1), that we can generate further solutions by increasing a by 8 and decreasing b by 19. We require $a+b<10$; our current solution gives $a+b=18$. The next solution is $a=31, b=-24$; this gives $a+b=7$, which is less than 10. Furthermore, subsequent solutions have larger values of a, and we require the smallest possible value of a. Thus, $a=31$ is the answer to the question.

T-5 Ellipse 1 lies in the first quadrant and is tangent to both axes; its major axis is parallel to the *y*-axis and its minor axis is parallel to the *x*-axis; the length of its semi-major axis is 2 and the length of its semi-minor axis is 1. Ellipse 2 has the same center as Ellipse 1, has the same eccentricity as Ellipse 1 (meaning that it is similar to Ellipse 1), and is internally tangent to Ellipse 1. The major axis of Ellipse 2 is perpendicular to the major axis of Ellipse 1. If the equation of Ellipse 2 is written as

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

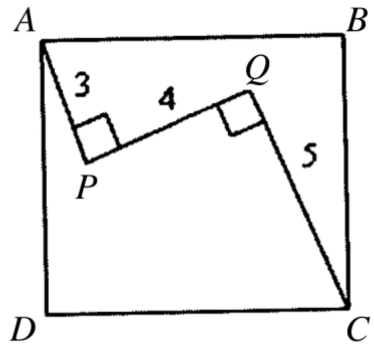
then $a+b+h+k=\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$.
[Answer: 11]

<u>Solution:</u>
Ellipse 1 has center (1,2) with y as a major axis of length 4, minor axis of length 2. Ellipse 2 has same center, major axis of length 2 and minor axis of length 1 to keep the same eccentricity and perpendicular to ellipse 1

resulting in the equation: $$\frac{(x-1)^2}{1^2}+\frac{(y-2)^2}{\frac{1}{2}^2}=1$$,

$1+2+1+\frac{1}{2}=\frac{9}{2},\ \ 9+2=11$

T-6 In the diagram, $ABCD$ is a square, $\overline{AP} \perp \overline{PQ}$, $\overline{PQ} \perp \overline{QC}$, $AP = 3$, $PQ = 4$, and $QC = 5$. Find the area of square $ABCD$.



[Answer: 40]

Solution:
Let the side length of the square be a and let the angle between line AP and line AD be θ. Then, looking at horizontal distances, $3\sin\theta + 4\cos\theta + 5\sin\theta = a$. Looking at vertical distances, $3\cos\theta - 4\sin\theta + 5\cos\theta = a$. Combining these two equations we get that $12\sin\theta = 4\cos\theta$, so $\tan\theta = 1/3$. Thus, $\sin\theta = 1/\sqrt{10}$ and $\cos\theta = 3/\sqrt{10}$. Returning to the first equation, we get $a = 8\sin\theta + 4\cos\theta$ $= 20/\sqrt{10} = 2\sqrt{10}$. So, the area of the square is $a^2 = 4 \cdot 10 = 40$.