Connecticut ARML Qualification Test, March 7, 2024

1. Find the last digit of $9^{8^{7}}$.
[Answer: 1]

2. Suppose the function $g(x) = f(x) + x^3$ is even and $f(-10) = 2024$. What is the value of $f(10)$?
[Answer: 24]

3. An equilateral triangle and a regular hexagon have equal perimeters. If the area of the triangle is 20 square units, what is the area of the hexagon?
[Answer: 30]

4. A sequence $\{a_n\}$ is defined by $a_{n+2} = 2a_{n+1} + a_n$ (for $n \geq 1$), and $a_6 = 181$, $a_7 = 437$. Find the value of a_4.
[Answer: 31]

5. Suppose that *a* and *b* are real numbers satisfying $\log_8 a^2 + \log_4 b^3 = 6$ and $\log_4 a^3 + \log_8 b^2 = 7$. Compute *ab*.
[Answer: 64]

6. Suppose that *pq* + *qr* = 243 for certain primes *p, q,* and *r*. Find *pqr*.
[Answer: 474]

7. Compute the sum of the roots of the polynomial
$(x + 1)^{2024} + (x - 2)^{2024} + (x + 3)^{2024} + (x - 4)^{2024} + \ldots + (x - 2024)^{2024}$.
[Answer: 1012]

8. Suppose that the following function is defined for $0 < x < \frac{\pi}{2}$. What is the minimum value of the function?

$$f(x) = \frac{1+\tan^2 x}{(\sin^3 x)(\cos x)}$$

[Answer: 8]

9. Find the remainder when $1! + 2! + 3! + \ldots + 100!$ is divided by 16.
[Answer: 9]

10. Let $\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{5^2}\right)\ldots\left(1-\frac{1}{98^2}\right)=\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
[Answer: 82]

11. When the number $222\ldots222_3$, with 1000 occurrences of the digit 2, is converted to base 9, the sum of the digits of the resulting base-9 number is S. Find S. (Your answer should be expressed in base 10.)
[Answer: 4000]

12. Alison, Justin, and Helen inherit their grandfather's flock of n emus. (Note: An emu is a type of bird.) According to the will, Alison is to receive $\frac{1}{2}$ of the emus, Justin is to receive $\frac{1}{3}$ of the emus, and Helen is to receive $\frac{1}{h}$ of the emus, where h is a positive integer. Unfortunately, n is not divisible by 2, 3, or h, and individual emus are not amenable to division, so the children are stuck until a neighbor gives them an emu. The increased flock is much easier to divide: Alison gets exactly $\frac{1}{2}$ of the emus, Justin gets exactly $\frac{1}{3}$ of the emus, and Helen gets exactly $\frac{1}{h}$ of the emus. Even better, there is exactly one emu left over, which they give back to the neighbor. Compute the greatest possible value of n.
[Answer: 41]

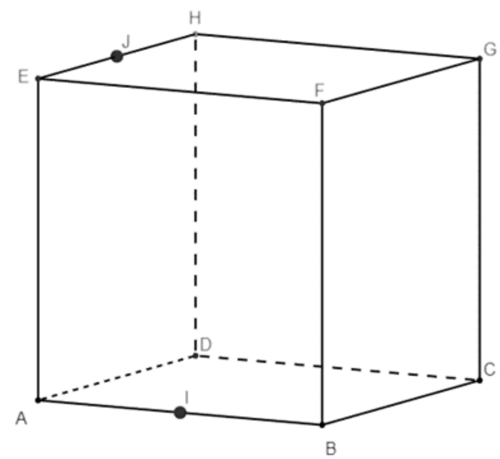
13. Circles of radius 5, 5, 8 and r are mutually externally tangent, where $r = m/n$ for relatively prime positive integers m and n. Find $m + n$.
[Answer: 17]

14. Three standard, fair, six-sided dice are rolled. Given that the sum of the values rolled is 11, the probability that none of the numbers showing is prime can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.
[Answer: 11]

15. A bag contains eight cards numbered 1, 2, 3, …, 8. Alicia, Brian, Chris, and Damon each take two cards from the bag without looking (and without replacing the cards), and add the numbers on their cards. The probability that all four sums are odd is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.
[Answer: 43]

16. $\left(\frac{\sqrt{6}+\sqrt{6}i}{\frac{3\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i}\right)^4$ is $a+bi$ for some real a, b. Find $a^2 + b^2$.
[Answer: 16]

17. An ant is crawling on the faces of a solid cube with side length 20. (See the diagram below.) Its path starts from the midpoint I of edge $\overline{AB}$ and ends at the midpoint J of edge $\overline{EH}$. Let the shortest distance that the ant needs to travel be d. Find the integer closest to d.



[Answer: 28]

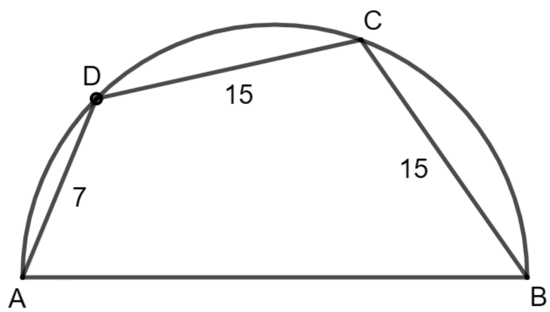
18. Rectangle $ABCD$ has $AB = 8$ and $BC = 6$. Triangle AEC is an isosceles right triangle with hypotenuse $\overline{AC}$ and E on the same side of $\overline{AC}$ as point B. Triangle BFD is an isosceles right triangle with hypotenuse $\overline{BD}$ and F on the same side of $\overline{BD}$ as point C. Find the area of the quadrilateral $BCFE$.

[Answer: 7]

19. Two ellipses, given by the equations $\frac{x^2}{2} + \frac{y^2}{4} = 1$ and $\frac{(x-1)^2}{9} + y^2 = 1$, intersect at four points. The center of the circle that passes through these four points is at $\left(\frac{1}{k}, 0\right)$. Find k.

[Answer: 17]

20. As shown in the diagram below, line segment AB is diameter of semicircle ω. Points C and D are on the semicircle with $AD = 7$, $CD = CB = 15$. Find the length of diameter AB.



[Answer: 25]

21. Five identical black marbles and seven identical white marbles are arranged in a line. How many distinguishable arrangements are there such that every black marble is next to a white marble?
[Answer: 356]

22. The roots of the polynomial $x^3 + px^2 + 24x + q$ are integers, not necessarily distinct. Suppose that 2 is one of the roots. Compute the greatest possible value of $|q|$.
[Answer: 180]

23. George has 3 chickens on his farm. Each day, each chicken has a ⅓ chance of escaping. Each chicken's behavior is independent of the behavior of the other chickens and independent of its own behavior on previous days. Let X be the whole number of days until all of George's chickens have escaped Then the expected value of X is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.
[Answer: 572]

24. A frog starts at the point (1, 1). Every second, if the frog is at point (x, y), it moves to $(x + 1, y)$ with probability $\frac{x}{x+y}$ and moves to $(x, y + 1)$ with probability $\frac{y}{x+y}$. The frog stops moving when its y-coordinate is 10. Let p be the probability that, when the frog stops, its x-coordinate is strictly less than 22. Then $p = \frac{m}{n}$, where m, n are relatively prime positive integers. Find $m + n$.
[Answer: 17]

25. For each subset $A \subseteq \{1, 2, ..., 20\}$, define $s(A) = \sum_{a \in A} a$, with $s(\phi) = 0$. Find the number of subsets A with $s(A) \equiv 0 \ (mod\ 5)$.
[Answer: 209728]